Naval Research Laboratory

Washington, DC 20375-5320

NRL/MR/5540--08-9160

The Bloch Sphere for Topologists

Ira S. Moskowitz

Center for High Assurance Computer Systems
Information Technology Division

November 5, 2008

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*
05-11-2008

2. REPORT TYPE
NRL Memorandum Report

3. DATES COVERED *(From - To)*
1 Jan 2008 – 15 Sept 2008

4. TITLE AND SUBTITLE
The Bloch Sphere for Topologists

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER
0602235N

6. AUTHOR(S)
Ira S. Moskowitz

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER
6326

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Research Laboratory, Code 5540
4555 Overlook Avenue, SW
Washington, DC 20375-5320

8. PERFORMING ORGANIZATION REPORT NUMBER
NRL/MR/5540--08-9160

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Office of Naval Research
One Liberty Center
875 North Randolph St., Suite 1425
Arlington, VA 22203-1995

10. SPONSOR / MONITOR'S ACRONYM(S)

11. SPONSOR / MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT
We use this note to clear up some of the mysteries of the Bloch sphere representation of pure states.

15. SUBJECT TERMS
Qubit, Quantum physics, Hopf fibration, Bloch sphere, Projective space

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
Unclassified	Unclassified	Unclassified	UL	12	Ira S. Moskowitz
					19b. TELEPHONE NUMBER *(include area code)* (202) 404-7930

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39.18

The Bloch Sphere for Topologists

Ira S. Moskowitz

Center for High Assurance Computer Systems
Code 5540
Naval Research Laboratory
Washington, DC 20375

Abstract

We use this note to clear up some of the mysteries of the Bloch sphere representation of pure states.

1. Introduction

The Bloch sphere is a representation of a pure state as a point on the unit sphere $S^2 \subset \mathbb{R}^3$. Pure state kets that are norm one scalar multiples of each other share the same representation. We shall make this clear.

We make no claims of originality in this note. Our purpose is to simply explain some of the basics of quantum information. For a general reference on this subject we recommend [6]. For the discussion of the Hopf fibration and the Bloch sphere we relied on [3, 8, 10, 11, 5].

We start with the two-dimensional Hilbert space $\mathbb{C}^2$ such that if $v = \binom{v_1}{v_2}$ and $w = \binom{w_1}{w_2}$ are elements of $\mathbb{C}^2$, then the inner product

$$v \cdot w = \overline{v_1} \cdot w_1 + \overline{v_2} \cdot w_2 = v^\dagger w \ .$$

Note the horizontal line over a (complex) number represents its conjugate, and also keep in mind that mathematicians often conjugate the second position, rather than the first as quantum physicists do. Note that the dual vector $v^\dagger$ is the conjugate transpose of v and the multiplication between $v^\dagger$ and w is matrix multiplication. The norm (length, magnitude) of v is $||v|| = \sqrt{v \cdot v}$. Elements of $\mathbb{C}^2$ are called *states*.

Next, we consider all the elements of $\mathbb{C}^2$ of length one. That set is simply the 3-sphere S^3 which is made up of elements of $v \in \mathbb{C}^2$ such that $||v|| = 1$. We call an element of S^3 under this construction a *pure state*. The elements $\binom{1}{0}$ and $\binom{0}{1}$ in $\mathbb{C}^2$ form an orthonormal basis of $\mathbb{C}^2$. They are also elements of S^3.

Manuscript approved September 24, 2008.

They are so special we give them special names: $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$. Here we have started using Dirac's [2] bra-ket notation for states. A general state is expressed as the ket $|\psi\rangle$ by

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

with $\alpha, \beta \in \mathbb{C}$. If we also have that $|\alpha|^2 + |\beta|^2 = 1$, then our state is a pure state. If a ket is a (complex) non-zero scalar multiple of another ket, those two kets represent the same physical state. (This is why pure state kets should really be viewed as elements of $\mathbb{C}P^1$ as explained later.) At this stage we could directly construct the Bloch sphere, but we choose to take a path that mimics what is explained in [6, Sec. 1.2]. We may normalize any $|\psi\rangle$ by $|\psi'\rangle = \frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}\,|\psi\rangle$. Physically, $|\psi\rangle$ and $|\psi'\rangle$ are the same. Note that this brings us from the 4-dimensional manifold $\mathbb{C}^2$, down a dimension, to the unit ball in 4-space, S^3. Pure state kets in S^3 are unique up to a (complex) scalar multiple of norm one. Using this thinking is how we will construct the Bloch sphere via the Hopf fibration from S^3 to $S^2 \cong \mathbb{C}P^1$.

$\mathbb{C}P^1$ is complex projective 1-space which is formed from $\mathbb{C}^2 - \binom{0}{0}$ by identifying points in $\mathbb{C}^2 - \binom{0}{0}$ that are the same up to a non-zero (complex) scalar multiple. So $\binom{z_1}{z_2} \sim \binom{w_1}{w_2}$ iff $z_1 = c \cdot w_1$ and $z_2 = c \cdot w_2$ for c a non-zero complex number. Hence we have the "mod" map E (sends an element to its equivalence class)

$$E : \mathbb{C}^2 - \binom{0}{0} \to \mathbb{C}P^1 \ .$$

Since $\mathbb{C}P^1$ is given the quotient topology, E is continuous and onto. We discuss this more later.

Since S^3 is a subset of $\mathbb{C}^2 - \binom{0}{0}$, we can also view E as a continuous map

$$E : S^3 \to \mathbb{C}P^1 \ .$$

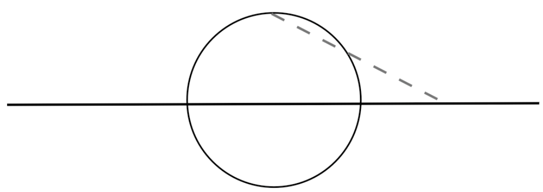
However, we can do better than that by taking any point $\binom{z_1}{z_2} \in \mathbb{C}^2 - \binom{0}{0}$ and mapping it to the point $\frac{1}{\sqrt{|z_1|^2+|z_2|^2}}\binom{z_1}{z_2} \in S^3$ via the map π to show that $E : S^3 \to \mathbb{C}P^1$ is also onto.

$$\begin{tikzcd}
\mathbb{C}^2 - \binom{0}{0} \arrow[d, "\pi"'] \arrow[dr, "E"] & \\
S^3 \arrow[r, "E"'] & \mathbb{C}P^1
\end{tikzcd}$$

The above diagram commutes and the bottom horizontal map is simply, as noted, the restriction of E to the subset $S^3 \subset \mathbb{C}^2 - \binom{0}{0}$.

In this paragraph we will show why we may view $\mathbb{C}P^1$ as $\mathbb{C}\cup\infty$, which is the one point compactification of the complex numbers. We denote the equivalence class of $\binom{z_1}{z_2}$ as $\left[\binom{z_1}{z_2}\right]$. We may uniquely identify any $\left[\binom{z_1}{z_2}\right]$, except for $\left[\binom{z_1}{0}\right]$ with a point in $\mathbb{C}$, by mapping $\left[\binom{z_1}{z_2}\right]$ to z_1/z_2, and by mapping $\left[\binom{z_1}{0}\right]$ to ∞. It is obvious that this mapping is well-defined. The mapping is onto because if $\zeta \in \mathbb{C}$, then $\left[\binom{\zeta}{1}\right]$ maps to ζ. Given $\binom{z_1}{z_2}$ we have that for any complex numbers w_1 and w_2 that $w_1 = \alpha z_1$ and that $w_2 = \beta z_2$, with $\alpha, \beta \in \mathbb{C} - 0$. The mapping is 1-1 because, unless $z_2 = 0$, if $\left[\binom{z_1}{z_2}\right] = \left[\binom{w_1}{w_2}\right]$ then $(\alpha/\beta)(z_1/z_2) = w_1/w_2 = z_1/z_2$ so $\alpha = \beta$. If $z_2 = 0$ then if the equivalence classes are the same we must have that w_2 is also zero, so there is no restriction on α. This (famous) bijective map from $\mathbb{C}P^1 \to \mathbb{C}\cup\infty$, along with its inverse, is continuous (not shown) and allows us to freely view $\mathbb{C}P^1$ as $\mathbb{C}\cup\infty$. Note that $\left[\binom{z_1}{z_2}\right] \to z_1/z_2$ is not the only "natural" homeomorphism that will work. In fact, we will actually use the conjugate mapping $\left[\binom{z_1}{z_2}\right] \to \overline{(z_1/z_2)}$ in our construction of the Hopf fibration.[1]

In this paragraph we describe a "natural" homeomorphism between S^2 and $\mathbb{C}P^1$ (taken as $\mathbb{C}\cup\infty$). View S^2 in the usual manner as being centered about the origin in $\mathbb{R}^3$, and view the subset $(x, y, 0)$ in $\mathbb{R}^3$ as $\mathbb{C}$. Draw a line from the north pole $(0, 0, 1)$ of S^2 to a point $\zeta \in \mathbb{C}$. Where this line hits S^2, other than the north pole, is how we uniquely identify ζ with a point on S^2. To extend the homeomorphism to the point at infinity we identify ∞ with the north pole itself.



We denote this above map by S

$$S : \mathbb{C}P^1 \to S^2$$

Therefore, by composing S with E and calling it H ($H = S \circ E$) we have

$$H : S^3 \to S^2 \ .$$

This map is simply the famous Hopf (circle) fibration of S^3 with base space S^2, first given in [3].

[1]The standard construction does not use the conjugate map. We use it so that our coordinates on the Bloch sphere are the classical spherical coordinates. However, the Hopf fibration "behaves" the same with either mapping.

2. Maps in detail

Let us flesh out the details of our maps. Recall, we are freely viewing $\mathbb{C}P^1$ as $\mathbb{C} \cup \infty$ under the map that takes the equivalence class of $\binom{z_1}{z_2}$ to the conjugate of z_1/z_2. We will start with E .

2.1. $E : S^3 \to \mathbb{C}P^1$

An element $\binom{z_1}{z_2}$of S^3 is an element of $\mathbb{C}^2 - \binom{0}{0}$ with the restriction that $|z_1|^2 + |z_2|^2 = 1$. The image of $\binom{z_1}{z_2}$ in $\mathbb{C}P^1$ under E is $\overline{(z_1/z_2)}$ and since $1/z_2 = \overline{z_2}/|z_2|^2$ and $\overline{\overline{z}} = z$, we have

$$E\left(\binom{z_1}{z_2}\right) = \overline{z_1} z_2/|z_2|^2 \ .$$

The infinities work out in E because z_1 and z_2 are never simultaneously zero and when dealing with infinities in complex variables a second order infinity "beats" a first order zero.

If $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$ then

$$E\left(\binom{x_1+ix_2}{x_3+ix_4}\right) = \frac{(x_1x_3 + x_2x_4) + i(x_1x_4 - x_2x_3)}{{x_3}^2 + {x_4}^2} \ .$$

2.2. $S : \mathbb{C}P^1 \to S^2$

If $\zeta = a + ib$ is a point in $\mathbb{C}$, with $\mathbb{C}$ being the plane $(x, y, 0) \subset \mathbb{R}^3$, then the coordinates of $\zeta \in \mathbb{R}^3$ are $(a, b, 0)$. The coordinates of the north pole are $(0, 0, 1)$. The straight line from the north pole to ζ is then given by parametrized vector in $\mathbb{R}^3$

$$\overrightarrow{V(t)} = (1 - t)(0, 0, 1) + t(a, b, 0)$$

We are interested in when this vector intersects S^2. Therefore, we solve

$$|\overrightarrow{V(t)}|^2 = 1 \ .$$

This gives us $t^2a^2+t^2b^2+(1-t)^2 = 1$ which has the solutions $t = 0$ or $t = 2/(1+a^2+b^2)$. We ignore the solution $t = 0$ because this is just the starting point-the north pole. So the point on S^3 that ζ maps to is $\frac{1}{1+a^2+b^2}(2a, 2b, a^2 + b^2 - 1)$. If ζ is in fact ∞ then S maps it to the north pole $(0, 0, 1)$.

2.3. $H : S^3 \to S^2$

Let $w = a + ib$ be the image of $\binom{x_1+ix_2}{x_3+ix_4}$ under E, so $a = \frac{x_1x_3+x_2x_4}{{x_3}^2+{x_4}^2}$ and $b = \frac{x_1x_4-x_2x_3}{{x_3}^2+{x_4}^2}$. So $a^2 + b^2 = \frac{{x_1}^2+{x_2}^2}{{x_3}^2+{x_4}^2}$, $1 + a^2 + b^2 = 1 + \frac{{x_1}^2+{x_2}^2}{{x_3}^2+{x_4}^2} = \frac{{x_1}^2+{x_2}^2+{x_3}^2+{x_4}^2}{{x_3}^2+{x_4}^2} = \frac{1}{{x_3}^2+{x_4}^2}$ since $\binom{x_1+ix_2}{x_3+ix_4} \in S^3$. We also have that $-1+a^2+b^2 = \frac{{x_1}^2+{x_2}^2-{x_3}^2-{x_4}^2}{{x_3}^2+{x_4}^2}$.

So $S(w) = \left(2(x_1x_3 + x_2x_4), 2(x_1x_4 - x_2x_3),\ {x_1}^2 + {x_2}^2 - {x_3}^2 - {x_4}^2\right)$. Combining everything gives us the Hopf map $H : S^3 \to S^2$ as follows:

$$H\left(\binom{x_1+ix_2}{x_3+ix_4}\right) = \left(2(x_1x_3 + x_2x_4), 2(x_1x_4 - x_2x_3),\ {x_1}^2 + {x_2}^2 - {x_3}^2 - {x_4}^2\right) .$$

Using the complex coordinates of an element of S^3 we can also write the Hopf map as:

$$H\left(\binom{z_1}{z_2}\right) = \left(2Re(\overline{z_1}z_2), 2Im(\overline{z_1}z_2),\ |z_1|^2 - |z_2|^2\right) . \tag{1}$$

If we view $\mathbb{R}^3$ as $\mathbb{C} \times \mathbb{R}$ we can also write the Hopf map as:

$$H\left(\binom{z_1}{z_2}\right) = \left(2\overline{z_1}z_2,\ |z_1|^2 - |z_2|^2\right) . \tag{2}$$

Let us show directly that the inverse image of H is made up of elements of S^3 that are the same up to scalar multiplication by a complex number of norm one.

Say $\binom{w_1}{w_2} = \lambda\binom{z_1}{z_2},\ |\lambda| = 1$. It is obvious that $\overline{\lambda z_1}\ \lambda z_2 = \overline{z_1}z_2$ and that $|\lambda|^2\left(|z_1|^2 + |z_2|^2\right) = |z_1|^2 + |z_2|^2$. Hence, $H\left(\lambda\binom{z_1}{z_2}\right) = H\left(\binom{z_1}{z_2}\right)$.

We know that H is onto since $E : S^3 \to \mathbb{C}P^1$ is. Say both $\binom{z_1}{z_2}$ and $\binom{w_1}{w_2}$ are in $H^{-1}(x_0, y_0, z_0)$ where $(x_0, y_0, z_0) \in S^2 \subset \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$. We know that $|w_1|^2 + |w_2|^2 = 1 = |z_1|^2 + |z_2|^2$. Therefore $|w_1|^2 = 1 - |w_2|^2$ and $|z_1|^2 = 1 - |z_2|^2$. Since H maps both $\binom{z_1}{z_2}$ and $\binom{w_1}{w_2}$ to (x_0, y_0, z_0), we have that $1 - 2|w_2|^2 = z_0 = 1 - 2|z_2|^2$, so $w_2 = \lambda_2 z_2, |\lambda_2| = 1$. Similarly, $w_1 = \lambda_1 z_1, |\lambda_1| = 1$. We also have that $2\overline{w_1}w_2 = x_0 + iy_0 = 2\overline{z_1}z_2$, which now gives us $\overline{\lambda_1}\lambda_2\overline{z_1}z_2 = \overline{z_1}z_2$. If $z_1 \neq 0 \neq z_2$, then $\overline{\lambda_1} = \overline{\lambda_2}/|\lambda_2|^2 = \overline{\lambda_2}$, so $\lambda_1 = \lambda_2$ as desired. If $z_2 = 0$ we must also have that $w_2 = 0$ since H maps them to the same point $(0, 0, 1) \in \mathbb{R}^3$. Therefore, $w_1 = \lambda z_1$ where $|\lambda|$ must be one (of course $0 = \lambda 0$). The same thing works if $z_1 = 0$.

So we have shown that the Hopf map $H : S^3 \to S^2$ is a map (in fact a smooth map) such that the inverse image of every point is (diffeomorphic to) the unit complex numbers (S^1 or equally $U(1)$). An obvious question is then — Is S^3 homeomorphic to $S^2 \times S^1$? The answer in no because S^3 is simply connected, where as $S^2 \times S^1$ is not. However, Hopf did show that a fiber bundle is formed via his map H. This is why it is called the Hopf *fibration* We explicitly show this below.

2.4. The Hopf fibration as a fiber bundle, $h : S^3 \to \mathbb{C}P^1$

To make matters simple take S^2 as $\mathbb{C}P^1$ and view S^3 as $(z_1, z_2) \in \mathbb{C}^2$ such that $|z_1|^2 + |z_2|^2 = 1$. Instead of using the map H as before, we use the map h which does not conjugate the ratio of the z_i. The reason we use H as given before was to be able to use the representation given in Eq. 2, which exactly gives us spherical coordinates for the Bloch sphere. If we were to use the map h we would still get a good representation, it just would not be classical spherical coordinates.

Let the open set $U \subset \mathbb{C}P^1$ be all of $\mathbb{C}P^1$ except for the point at infinity (north pole of S^2). Then $h^{-1}(U)$ is an open set in S^3. We wish to show that (1)-the following diagram commutes (where $proj_1$ is projection onto the first factor), and (2) the map f is a homeomorphism.

$$\begin{tikzcd} h^{-1}(U) \subset S^3 \arrow[r, "f"] \arrow[d, "h"'] & U \times S^1 \subset \mathbb{C}P^1 \times S^1 \arrow[dl, "proj_1"] \\ U \subset \mathbb{C}P^1 & \end{tikzcd}$$

If $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$, set $f\left((z_1, z_2)\right) = (z_1/z_2, e^{i\phi_2})$. This is well-defined since $z_2 \neq 0$. The inverse map of f is $f^{-1}(\zeta, e^{i\theta}) = \frac{1}{\sqrt{1+|\zeta|^2}}(e^{i\theta}\zeta, e^{i\theta})$. If we write the z_i as $r_i e^{i\phi_i}$ and write $|\zeta|^2$ as $\frac{{r_1}^2}{{r_2}^2}$ and use the fact that ${r_1}^2 + {r_2}^2 = 1$ we see that f^{-1} *is* the inverse of f.

We now let V be the open subset of $\mathbb{C}P^1$ that is everything except for the origin (south pole). Then $h^{-1}(V)$ is an open set in S^3 we wish to show that (1)-the following diagram commutes (where $proj_1$ is projection onto the first factor), and (2) the map f' is a homeomorphism.

$$\begin{tikzcd} h^{-1}(V) \subset S^3 \arrow[r, "f'"] \arrow[d, "h"'] & V \times S^1 \subset \mathbb{C}P^1 \times S^1 \arrow[dl, "proj_1"] \\ V \subset \mathbb{C}P^1 & \end{tikzcd}$$

If $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$, set $f'\left((z_1, z_2)\right) = (z_1/z_2, e^{i\phi_1})$. This is well-defined since $z_1 \neq 0$. The inverse map of f' is ${f'}^{-1}(\zeta, e^{i\theta}) = \frac{1}{\sqrt{1+|\zeta^{-1}|^2}}(e^{i\theta}, e^{i\theta}\zeta^{-1})$.

Thus, we have shown that we have a fiber bundle, since locally we can show that S^3 is homeomorphic to the cross-product $\mathbb{C}P^1 \times S^1$ and the local homeomorphisms respect the Hopf map.

3. Implicit Function Theorem

We end this note by pointing out that we could have quickly shown that the inverse image in S^3 of every point on S^2 is a circle. The Hopf map from S^3 to S^2 is a submersion. By the implicit function theorem $H^{-1}(s)$ is a 1-manifold. Since s is closed H^{-1} must be a closed 1-manifold, which means that it is homeomorphic to a circle. Of course this view does not prove that the Hopf fibration is actually a fiber bundle but we could show that by viewing S^1 as the unitary group $U(1)$ and letting $U(1)$ act on S^3 by scalar multiplication.

$S^3/U(1) \cong S^2$ with a resulting fiber bundle structure [9, Sec. 20]. In fact, the fibration of S^3 by circles is also a codimension-2 foliation of S^3 [4, 7]. We chose not to go this way in explaining the Bloch representation of pure state kets (qubits) since the less machinery used the better.

3.1. Intuitiveness

We start off with the Hilbert space $\mathbb{C}^2$. Forgetting about the inner product stuff, we see that this is just the 4-dimensional manifold $\mathbb{R}^4$. The scalar multiplication in the Hilbert space allows us to multiply a ket by any complex scalar. A complex scalar $re^{i\theta}$ itself has two degrees of freedom, one is r and the other is θ. By normalizing the ket we are loosing the degree of freedom from r, hence we drop a dimension from four down to three—this is exactly what happens as we go from $\mathbb{C}^2$ to S^3. Now we are left with one degree of freedom θ. When we ignore θ we loose another dimension and drop down to two; this is how we end up with S^2. It would be interesting to see if instead of looking at the circle action on S^3 if we could look at it on $\mathbb{C}^2$ to begin with and then normalize by r.

4. Spherical Coordinates

Given any pure state ket $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, we can express the Hopf map on $|\psi\rangle$ as $H(|\psi\rangle) = H\left(\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)\right) = (2\overline{\alpha}\beta, |\alpha|^2 - |\beta|^2) \in \mathbb{C}\times\mathbb{R} \cong \mathbb{R}^3$, which is justified since $|0\rangle = \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$ and $|1\rangle = \left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)$ both as points in S^3, which are also points in $\mathbb{C}^2$. If $\alpha = r_\alpha e^{i\theta_\alpha}$ and $\beta = r_\beta e^{i\theta_\beta}$, consider $e^{-i\theta\alpha}\,|\psi\rangle = |\psi'\rangle = r_\alpha\,|0\rangle + r_\beta e^{i(\theta_\beta - \theta_\alpha)}\,|1\rangle$. Since $H(|\psi\rangle) = H(|\psi'\rangle)$, we need only analyze pure state kets of the form $r_\alpha\,|0\rangle + r_\beta e^{i\phi}\,|1\rangle, \phi \in [0, 2\pi)$. Since we are dealing with pure state kets we know that $(r_\alpha)^2 + (r_\beta)^2 = 1$, By adjusting r_α we see that r_β can be any value between zero and one. Therefore, we can[2] set $r_\beta = \sin(\theta/2)$, for $\theta \in [0, \pi]$. This forces $r_\alpha = \cos(\theta/2)$.

Hence (as in [6, Eq. (1.3)], which made no mention of the Hopf fibration) we can express, for purposes of the Hopf map, any $|\psi\rangle = e^{i\gamma}\left(\cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle\right) \in S^3$ as

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle \ .$$

Viewing $|\psi\rangle \in S^3$, we write it as

$$|\psi\rangle = \begin{pmatrix}\cos(\theta/2)\\ e^{i\phi}\sin(\theta/2)\end{pmatrix} , \tag{3}$$

Let us apply the Hopf map as given in Eq. (2):

$$H\left(\begin{pmatrix}\cos(\theta/2)\\ e^{i\phi}\sin(\theta/2)\end{pmatrix}\right) = \left(2e^{i\phi}\cos(\theta/2)\sin(\theta/2), \cos^2(\theta/2) - \sin^2(\theta/2)\right) \ .$$

[2] The reason for the odd choice of $\theta/2, \theta \in [0, \pi]$ instead of the simpler $\theta, \theta \in [0, \pi/2]$ will become clear later.

However, by Eq. (1) this is

$$\big(2\cos\phi\cos(\theta/2)\sin(\theta/2), 2\sin(\phi)\cos(\theta/2)\sin(\theta/2), \cos^2(\theta/2)-\sin^2(\theta/2)\big) \in \mathbb{R}^3. \tag{4}$$

This reduces to the spherical coordinates

$$(x, y, z) = (\cos\phi\sin\theta,\ \sin\phi\sin\theta,\ \cos\theta)\,, \tag{5}$$

$\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$, which up to a swap of ϕ and θ, are the classical spherical coordinates.

5. Density Operators for Pure States

Given any pure state[3] ket $\alpha|0\rangle + \beta|1\rangle$ we can form the outer product $|\psi\rangle\langle\psi|$ which is

$$\binom{\alpha}{\beta}\binom{\alpha}{\beta}^{\dagger} = \begin{pmatrix} |\alpha|^2 & \alpha\overline{\beta} \\ \overline{\alpha}\beta & |\beta|^2 \end{pmatrix}. \tag{6}$$

Notice that if $|\lambda| = 1$, then $|\lambda\psi\rangle\langle\lambda\psi| = |\psi\rangle\langle\psi|$.

Using the representation of $|\psi\rangle$ as given in Eq. (3) and the identities $2\cos^2(\theta/2) - 1 = \cos\theta$ we see that $|\alpha|^2 = \cos^2(\theta/2) = \frac{1}{2}(1+\cos\theta)$. Using Eq. (5) then we see that $|\alpha|^2 = \frac{1}{2}(1+z)$. Using the identity $1-2\sin^2(\theta/2) = \cos(\theta)$, the representation Eq. (3) and Eq. (5) we similarly have that $|\beta|^2 = \frac{1}{2}(1-z)$. In the same manner we have $\overline{\alpha}\beta = \frac{1}{2}e^{i\phi}\sin\theta = \frac{1}{2}(x+iy)$ and $\alpha\overline{\beta} = \frac{1}{2}e^{-i\phi}\sin\theta = \frac{1}{2}(x-iy)$. This allows us to express $|\psi\rangle\langle\psi|$ also as

$$|\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}. \tag{7}$$

Since the the density operator mapping $|\psi\rangle \to |\psi\rangle\langle\psi|$ is invariant under multiplying $|\psi\rangle$ by a norm one scalar, if we restrict our kets to S^3 we can take the density operator mapping as an alternate Hopf map by using Eq. (6) to go back to Eq. (5), and then back to Eq. (2). Of course, there is no a priori reason to, in general, restrict the density operator mapping to S^3 it is perfectly valid on the Hilbert space $\mathbb{C}^2$.

It is worth pointing out that if one had no knowledge of the Hopf map and only Eq. (6), and guessed that Eq. (6) was the same as Eq. (7) (that is $x+iy = 2\overline{\alpha}\beta$ and $z = |\alpha|^2 - |\beta|^2$), they would then have the Hopf map by interpreting x, y, and z as points in $\mathbb{R}^3$ and seeing that $x^2+y^2+z^2 = 1$.

The challenge now is to see how algebraic topology can help us with mixed states.

[3]The kets need not be pure state to obtain the outer product matrix, but we require it none the less.

6. Pauli Matrices

By using the Pauli matrices [6], one can naturally [1] derive the spherical coordinates from Eq. (7).

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I} \tag{8}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{9}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{10}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{11}$$

We follow the definition for the Hopf map as given in [1]. For $\psi = \binom{\alpha}{\beta} \in S^3$, consider:

$$\langle\psi|\, \sigma_x \,|\psi\rangle, \langle\psi|\, \sigma_y \,|\psi\rangle, \text{ and } \langle\psi|\, \sigma_z \,|\psi\rangle \ . \tag{12}$$

We easily see that

$$\sigma_x \,|\psi\rangle = \sigma_x \binom{\alpha}{\beta} = \binom{\beta}{\alpha} \text{ so,} \tag{13}$$

$$\langle\psi|\, \sigma_x \,|\psi\rangle = \overline{\alpha}\beta + \alpha\overline{\beta} \tag{14}$$

$$\sigma_y \,|\psi\rangle = \sigma_y \binom{\alpha}{\beta} = \binom{-i\beta}{i\alpha} \text{ so,} \tag{15}$$

$$\langle\psi|\, \sigma_y \,|\psi\rangle = -i\overline{\alpha}\beta + i\alpha\overline{\beta} \text{ and} \tag{16}$$

$$\sigma_z \,|\psi\rangle = \sigma_z \binom{\alpha}{\beta} = \binom{\alpha}{-\beta} \text{ so,} \tag{17}$$

$$\langle\psi|\, \sigma_z \,|\psi\rangle = |\alpha|^2 - |\beta|^2 \ . \tag{18}$$

But we have that $2Re(\overline{\alpha}\beta) = \overline{\alpha}\beta + \alpha\overline{\beta}$, and that $2Im(\overline{\alpha}\beta) = -i\overline{\alpha}\beta + i\alpha\overline{\beta}$. Thus, we see that the Hopf map Eq. (2), with pure state $|\psi\rangle = \binom{\alpha}{\beta}$ may be rewritten as:

$$H(\psi) = (\langle\psi|\, \sigma_x \,|\psi\rangle, \langle\psi|\, \sigma_y \,|\psi\rangle, \langle\psi|\, \sigma_z \,|\psi\rangle) \ . \tag{19}$$

Consider an arbitrary 2×2 matrix $M = \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix}$ and pure state outer product outer product density matrix $|\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\overline{\beta} \\ \overline{\alpha}\beta & |\beta|^2 \end{pmatrix}$. Trivial calculations show that

$$\langle\psi|\, M \,|\psi\rangle = tr\Big(M \,|\psi\rangle\langle\psi|\Big) \ . \tag{20}$$

So, if $\rho = |\psi\rangle\langle\psi|$, we may write the Hopf map as

$$H(\psi) = \Big(tr(\sigma_x \rho), tr(\sigma_y \rho), tr(\sigma_z \rho) \Big) \,. \tag{21}$$

The above Eq. (21) gives us hope of generalizing the Hopf map to mixed states and seeing if that generalization corresponds to the known results on the Bloch ball [6, Ex. 2.72].

7. Acknowledgements

We are grateful to Keye Martin for the use of his 2008 summer interns, Philip Brunetti and Jacob Farinholt, as a receptive audience for my topology lectures. We also thank Tanner Crowder for his comments.

References

[1] Dariuz Chruściński. Geometric aspects of quantum mechanics and quantum entanglement. *Journal of Physics: Conference Series*, 30:9–16, 2006.

[2] P.A.M. Dirac. *The Principles of Quantum Mechanics.* Oxford, 1958.

[3] Heinz Hopf. Über die abbildungen der dreidimensionalen sphäre auf die kugelfläche. *Mathematische Annalen*, 104:637–665, 1931.

[4] H. Blaine Lawson. Foliations. *Bulletin of the American Mathematical Society*, 80(3):369–418, May 1931.

[5] Rémy Mosseri and Rossen Dandoloff. Geometry of entangled states, Bloch spheres and Hopf fibrations. *J. Phys. A: Math. Gen.*, 34:10243–10252, 2001.

[6] Michale A. Nielsen and Issac L. Chuang. *Quantum Computation and Quantum Information.* Cambridge, 2000.

[7] Seiki Nishikaa, Philippe Tondeur, and Lieven Vanhecke. Spectral geometry for Riemmanian foliations. *Annals of Global Analysis and Gemetry*, 10:291–304, 1992.

[8] Edwin H. Spanier. *Algebraic Topology.* McGraw-Hill, 1966.

[9] Norman Steenrod. *The Topology of Fibre Bundles.* Princeton, 1974.

[10] H. Urbantke. Two-level quantum systems: States, phases, and holonomy. *Am. J. Phys.*, 59(6):503–508, June 1991.

[11] H. Urbantke. The Hopf fibration — seven times in physics. *Journal of Geometry and Physics.*, 46:125–150, 2003.